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# Space-like submanifolds in de Sitter spaces

**Ximin Liu**

Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, People's Republic of China

E-mail: xmliu@dlut.edu.cn

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**Abstract**
Let $M^n$ be a space-like submanifold in a de Sitter space $M_p^{n+p}(c)$ with flat normal bundle. This paper gives some intrinsic conditions for $M^n$ to be totally umbilical.

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## 1. Introduction

Let $M_p^{n+p}(c)$ be an $(n + p)$-dimensional connected semi-Riemannian manifold of constant curvature $c$ whose index is $p$. It is called an indefinite space form of index $p$ and simply a space form when $p = 0$. If $c > 0$, we call it a de Sitter space of index $p$. Akutagawa [2] and Ramanathan [11] investigated space-like hypersurfaces in a de Sitter space and proved independently that a complete space-like hypersurface in a de Sitter space with constant mean curvature is totally umbilical if the mean curvature $H$ satisfies $H^2 \leqslant c$ when $n = 2$ and $n^2H^2 < 4(n-1)c$ when $n \geqslant 3$. Later, Cheng [4] generalized this result to general submanifolds in a de Sitter space.

To our knowledge, there were almost no intrinsic rigidity results for the space-like submanifolds with constant scalar curvature in a de Sitter space until Zheng [12] obtained the following result (see also [13] for a weak version of this result): let $M^n$ be an $n$-dimensional compact space-like hypersurface in $M_1^{n+1}(c)$ with constant scalar curvature. If $M^n$ satisfies $K(M) \geqslant 0$, $R < c$, where $R$ is the normalized scalar curvature of $M^n$, then $M^n$ is totally umbilical.

In [6], Cheng and Yau firstly studied the rigidity problem for a hypersurface with constant scalar curvature in a space form by introducing a self-adjoint second-order differential operator $\Box$. They proved that, for an $M^n$ in a space form $M^{n+1}(c)$, if $R$ is constant and $R \geqslant c$, then $|\nabla h|^2 \geqslant n^2|\nabla H|^2$ where $h$ and $H$ denote the second fundamental form and the length of the mean curvature vector field of $M^n$ respectively. By using Cheng and Yau's technique, Cheng and Ishikawa [5] have recently shown that the totally umbilical round spheres are the only compact space-like hypersurfaces in $S_1^{n+1}(1)$ with constant scalar curvature $S < n(n-1)$. Some other authors, such as Liu [9] and Li [8], have also obtained interesting results related to the characterization of the totally umbilical round spheres as the only compact space-like hypersurfaces in the de Sitter space with constant scalar curvature.

In this paper, we extend Cheng and Yau's technique to higher codimensional cases and use their operator $\Box$ to study the rigidity problem for space-like submanifolds in a de Sitter space with flat normal bundle.

## 2. Preliminaries

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional semi-Riemannian manifold of constant curvature $c$ whose index is $p$. Let $M^n$ be an $n$-dimensional Riemannian manifold immersed in $M_p^{n+p}(c)$. As the semi-Riemannian metric of $M_p^{n+p}(c)$ induces the Riemannian metric of $M^n$, $M^n$ is called a space-like submanifold. We choose a local field of semi-Riemannian orthonormal frames $e_1, \ldots, e_{n+p}$ in $M_p^{n+p}(c)$ such that at each point of $M^n$, $e_1, \ldots, e_n$ span the tangent space of $M^n$ and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leqslant A, B, C, \ldots \leqslant n+p \qquad 1 \leqslant i, j, k, \ldots \leqslant n \qquad n+1 \leqslant \alpha, \beta, \gamma \leqslant n+p.$$

Let $\omega_1, \ldots, \omega_{n+p}$ be its dual frame field so that the semi-Riemannian metric of $M_p^{n+p}(c)$ is given by $\mathrm{d}\bar{s}^2 = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \epsilon_A \omega_A^2$, where $\epsilon_i = 1$ and $\epsilon_\alpha = -1$. Then the structure equations of $M_p^{n+p}(c)$ are given by

$$\mathrm{d}\omega_A = \sum_B \epsilon_B \omega_{AB} \wedge \omega_B \qquad \omega_{AB} + \omega_{BA} = 0 \tag{1}$$

$$\mathrm{d}\omega_{AB} = \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \tfrac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D \tag{2}$$

$$K_{ABCD} = c\, \epsilon_A \epsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{3}$$

Restricting these forms to $M^n$, we have

$$\omega_\alpha = 0 \qquad n+1 \leqslant \alpha \leqslant n+p. \tag{4}$$

The Riemannian metric of $M^n$ is written as $\mathrm{d}s^2 = \sum_i \omega_i^2$. From Cartan's lemma we can write

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j \qquad h_{ij}^\alpha = h_{ji}^\alpha. \tag{5}$$

From these formulae, we obtain the structure equations of $M^n$:

$$\mathrm{d}\omega_i = \sum_j \omega_{ij} \wedge \omega_j \qquad \omega_{ij} + \omega_{ji} = 0 \tag{6}$$

$$\mathrm{d}\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \tfrac{1}{2} \sum_{k,l} K_{ijkl} \omega_k \wedge \omega_l \tag{7}$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha) \tag{8}$$

where $R_{ijkl}$ are the components of the curvature tensor of $M^n$.

For details on indefinite Riemannian manifolds see O'Neill [10]. We call

$$h = \sum_\alpha h_\alpha e_\alpha = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha \tag{9}$$

the second fundamental form of $M^n$ and the square length of the second fundamental form is defined by

$$S = \sum_\alpha \operatorname{tr}(h_\alpha)^2 = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2 = |h|^2. \tag{10}$$

The mean curvature vector $\xi$ of $M^n$ is defined by

$$\xi = \frac{1}{n} \sum_\alpha \operatorname{tr}(h_\alpha) e_\alpha = \frac{1}{n} \sum_\alpha \left( \sum_i h_{ii}^\alpha \right) e_\alpha \tag{11}$$

and we know that $\xi$ is independent of the choice of unit normal vectors $e_{n+1}, \ldots, e_{n+p}$ to $M^n$. The length of the mean curvature vector is called the mean curvature of $M^n$, denoted by $H$. From now on we assume that $\xi \neq 0$ and we choose the first unit normal vector $e_{n+1}$ to $M^n$ in the direction $\xi$. Therefore, we have

$$H = \frac{1}{n} \operatorname{tr}\, h_{n+1} = \frac{1}{n} \sum_i h_{ii}^{n+1} > 0 \tag{12}$$

$$\operatorname{tr}\, h_\alpha = \sum_i h_{ii}^\alpha = 0 \qquad \alpha = n+2, \ldots, n+p. \tag{13}$$

If there exist $p$ functions $\rho_\alpha$ such that $h_{ij}^\alpha = \rho_\alpha \delta_{ij}$ at each point of $M^n$, we call $M^n$ a totally umbilical submanifold. For a totally umbilical submanifold, we have

$$\rho_\alpha = \frac{1}{n} \operatorname{tr}\, h_\alpha = \frac{1}{n} \sum_i h_{ii}^\alpha. \tag{14}$$

Let $h_{ijk}^\alpha$ and $h_{ijkl}^\alpha$ denote the covariant derivative and the second covariant derivative of $h_{ij}^\alpha$, respectively. Then we have $h_{ijk}^\alpha = h_{ikj}^\alpha$ which implies

$$h_{ijkm}^\alpha = h_{ikjm}^\alpha \tag{15}$$

$$h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_m h_{im}^\alpha R_{mjkl} + \sum_m h_{jm}^\alpha R_{mikl} + \sum_\beta h_{ij}^\beta R_{\alpha\beta kl} \tag{16}$$

where $R_{\alpha\beta kl}$ are the components of the normal curvature tensor of $M^n$, that is

$$R_{\alpha\beta kl} = \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta). \tag{17}$$

If $R_{\alpha\beta kl} = 0$ at point $x$ of $M^n$ we say that the normal connection of $M^n$ is flat at $x$ and it is well known [3] that $R_{\alpha\beta kl} = 0$ at $x$ if and only if $h_\alpha$ are simultaneously diagonalizable at $x$.

## 3. Space-like submanifolds with flat normal bundle

Cheng and Yau [6] gave a lower estimation for $|\nabla h|^2$, the square of the length of the covariant derivative of $h$, which plays an important role in their discussion. They proved that, for a hypersurface in a space form of constant scalar curvature $c$, if the normalized scalar curvature $R$ is constant and $R \geqslant c$, then $|\nabla h|^2 \geqslant n^2 |\nabla H|^2$.

For the space-like submanifolds in a de Sitter space, we can prove the following theorem.

**Theorem 3.1.** *Let $M^n$ be a compact space-like submanifold in $M_p^{n+p}(c)$ with nowhere zero mean curvature $H$. If $R$ is constant and $R < c$, then*

$$|\nabla h|^2 = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 \geqslant n^2 |\nabla H|^2. \tag{18}$$

*Moreover, if the equality in (18) holds on $M^n$, then $H$ is constant.*

**Proof.** From (8), we have $n^2H^2 - |h|^2 = n(n-1)(c-R) > 0$. Taking the covariant derivative on both sides of this equality, we get

$$n^2H\,H_k = \sum_{i,j,\alpha} h_{ij}^{\alpha} h_{ijk}^{\alpha} \qquad k = 1, \dots, n.$$

For every $k$, it follows from the Cauchy–Schwarz inequality that

$$n^4H^2H_k^2 = \left(\sum_{i,j,\alpha} h_{ij}^{\alpha} h_{ijk}^{\alpha}\right)^2 \leqslant |h|^2 \sum_{i,j,\alpha} (h_{ijk}^{\alpha})^2 \tag{19}$$

where the equality holds if and only if there exits a real function $c_k$ such that

$$h_{ijk}^{\alpha} = c_k\, h_{ij}^{\alpha} \tag{20}$$

for all $i$, $j$ and $\alpha$. Summing on both sides of (19) with respect to $k$, we have

$$n^4H^2|\nabla H|^2 = n^4H^2\sum_k H_k^2 \leqslant |h|^2 \sum_{(i,j,k,\alpha)} (h_{ijk}^{\alpha})^2 \leqslant n^2H^2 \sum_{(i,j,k,\alpha)} (h_{ijk}^{\alpha})^2. \tag{21}$$

Therefore (18) holds on $M^n$.

Suppose that $|\nabla h|^2 = n^2|\nabla H|^2$ holds on $M^n$. It follows from (21) that

$$0 \leqslant n^3(n-1)(c-R)|\nabla H|^2 \leqslant |h|^2 \left(\sum_{i,j,k,\alpha} (h_{ijk}^{\alpha})^2 - n^2|\nabla H|^2\right). \tag{22}$$

Hence $(c-R)|\nabla H|^2 = 0$ on $M^n$. Because $R < c$, $|\nabla H|^2 = 0$ on $M^n$, hence $H$ is constant on $M^n$. This completes the proof of theorem 3.1. □

In the following, we propose to use Cheng and Yau's operator □ to study the rigidity problem for compact space-like submanifolds in the de Sitter space $M_p^{n+p}(c)$.

We know that the Laplacian $\Delta h_{ij}^{\alpha}$ of the fundamental form $h_{ij}^{\alpha}$ is defined to be $\sum_k h_{ijkk}^{\alpha}$, and hence, using (15), (16) and the assumption that $M^n$ has flat normal bundle, we have

$$\begin{aligned}\Delta h_{ij}^{\alpha} &= \sum_k (h_{ijkk}^{\alpha} - h_{ikjk}^{\alpha}) + \sum_k (h_{ikjk}^{\alpha} - h_{ikkj}^{\alpha}) + \sum_k (h_{ikkj}^{\alpha} - h_{kkij}^{\alpha}) + (\mathrm{tr}\ h_{\alpha})_{ij} \\ &= \sum_{m,k} h_{im}^{\alpha} R_{mkjk} + \sum_{m,k} h_{mk}^{\alpha} R_{mijk} + (\mathrm{tr}\ h_{\alpha})_{ij}\end{aligned} \tag{23}$$

where $(\mathrm{tr}\ h_{\alpha})_{ij}$ denotes the second covariant derivative of $(\mathrm{tr}\ h_{\alpha})$. Since the normal bundle of $M^n$ is flat, we choose $e_1, \dots, e_n$ such that

$$h_{ij}^{\alpha} = \lambda_i^{\alpha}\delta_{ij} \qquad \alpha = n+1, \dots, n+p. \tag{24}$$

Then the Laplacian of $|h|^2 = \sum_{i,j,\alpha} (h_{ij}^{\alpha})^2$ is given by

$$\begin{aligned}\tfrac{1}{2}\Delta|h|^2 = \tfrac{1}{2}n^2\Delta H^2 &= |\nabla h|^2 + \sum_{i,j,\alpha} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} \\ &= |\nabla h|^2 + n\sum_i \lambda_i^{n+1} H_{ii} + \tfrac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^{\alpha} - \lambda_j^{\alpha})^2.\end{aligned} \tag{25}$$

We define an operator □ acting on $f$ by

$$\Box f = \sum_{i,j} (nH\delta_{ij} - h_{ij}^{n+1}) f_{ij}. \tag{26}$$

Since $(nH\delta_{ij} - h_{ij}^{n+1})$ is trace-free it follows from [6] that the operator □ is self-adjoint relative to the $L^2$-inner product of $M^n$, i.e.

$$\int_{M^n} f\cdot\Box g = \int_{M^n} g\cdot\Box f. \tag{27}$$

Thus we have

$$\square H = \sum_{i,j}(nH\delta_{ij} - h_{ij}^{n+1})H_{ij} = nH\sum_i H_{ii} - \sum_i \lambda_i^{n+1}H_{ii}$$
$$= \tfrac{1}{2}n(\Delta H^2 - 2|\nabla H|^2) - \sum_i \lambda_i^{n+1}H_{ii}. \tag{28}$$

Hence we can prove the following theorem.

**Theorem 3.2.** *Let $M^n$ be a compact space-like submanifold with non-negative sectional curvature in $M_p^{n+p}(c)$. Suppose that the normal bundle $N(M)$ is flat and the normalized mean curvature vector is parallel. If $R$ is constant and $R < c$, then $M^n$ is totally umbilical.*

**Proof.** From (25) and (28), we have

$$n\square H = |\nabla h|^2 - n^2|\nabla H|^2 + \tfrac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2. \tag{29}$$

Since $\square$ is self-adjoint, we conclude that

$$0 \geqslant \int_{M^n}\left\{(|\nabla h|^2 - n^2|\nabla H|^2) + \tfrac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2\right\}. \tag{30}$$

Thus, by hypothesis and theorem 3.1, we have $|\nabla h|^2 = n^2|\nabla H|^2$, and $H$ is constant on $M^n$, then $\xi$ is parallel. Hence, our theorem follows immediately from a result of Aiyama [1, theorem 3] and this completes the proof of the theorem 3.2. □

**Remark 3.1.** It should be pointed out that the assumption that the normalized mean curvature vector field $\xi/H$ of $M^n$ is parallel is different from assumption that the mean curvature vector field $\xi$ of $M^n$ is parallel. It is significant to consider the difference. Li [7] has proven that, if $M^n$ is a closed and oriented pseudo-umbilical submanifold in a space form and $H$ is nowhere zero, them $H$ is constant if and only if $\xi/H$ is parallel.

**Theorem 3.3.** *Let $M^n$ be a compact space-like submanifold with non-negative sectional curvature in $M_p^{n+p}(c)$. Suppose that $M^n$ has flat normal bundle, if the normalized scalar curvature $R$ of $M^n$ is proportional to the mean curvature $H$ of $M^n$, that is $R = aH$, where $a$ is any constant. Then $M^n$ is totally umbilical.*

**Proof.** From (8) and the hypothesis, we have

$$|h|^2 = n^2H^2 - n(n-1)(c - aH). \tag{31}$$

Taking the covariant derivative of (31), we have for each $k$

$$(2n^2H + n(n-1)a)H_k = \sum_{i,j,\alpha} h_{ij}^\alpha h_{ijk}^\alpha$$

and hence, by the Cauchy–Schwarz inequality, we have

$$(2n^2H + n(n-1)a)^2|\nabla H|^2 \leqslant 4\sum_{i,j,\alpha}(h_{ij}^\alpha)^2\sum_{i,j,k,\alpha}(h_{ijk}^\alpha)^2$$

that is

$$(2n^2H + n(n-1)a)^2|\nabla H|^2 \leqslant 4|h|^2|\nabla h|^2. \tag{32}$$

From (31) and (32), we have

$$|\nabla h|^2 - n^2|\nabla H|^2 \geqslant |h|^{-2}\left[\frac{(2n^2H + n(n-1)a)^2}{4} - n^2|h|^2\right]|\nabla H|^2$$
$$= |h|^{-2}n^2(n-1)\left(\frac{(n-1)a^2}{4} + nc\right)|\nabla H|^2. \tag{33}$$

So from (30) and (33), we have

$$0 \geqslant \int_{M^n} \left\{ |h|^{-2} n^2 (n-1) \left( \frac{(n-1)a^2}{4} + nc \right) |\nabla H|^2 + \frac{1}{2} \sum_{i,j,\alpha} R_{ijij} (\lambda_i^\alpha - \lambda_j^\alpha)^2 \right\}. \tag{34}$$

Thus, by hypothesis, $|\nabla H|^2 = 0$, so $H$ is constant on $M^n$, hence $\xi$ is parallel, as from theorem 3 of [1] we know that $M^n$ is totally umbilical. □

**Remark 3.2.** In theorems 3.2 and 3.3, we have used assumptions that are different from that in [1, theorem 3] to obtain the same result.

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